

Tutorium Mathe 2 MT

Aufgabenblatt: Integralrechnung / Gebietsintegrale (+ Lösungen)

- 1) Berechnen Sie das Gebietsintegral $\iint_{0 \leq y \leq x \leq 1} (x^2 + y^2) dx dy$.

$$\begin{aligned} \int_0^1 \left(\int_0^x x^2 + y^2 dy \right) dx &= \int_0^1 x^2 y + \frac{y^3}{3} \Big|_0^x dx \\ &= \frac{4}{3} \int_0^1 x^3 dx \\ &= \frac{4}{3} \cdot \frac{x^4}{4} \Big|_0^1 \\ &= \frac{4}{3} \cdot \frac{1}{4} \\ &= \frac{1}{3} \end{aligned}$$

- 2) Berechnen Sie das Integral

$$\iint_D (1 - 2x - 3y) dx dy$$

über dem Gebiet $D := \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{\pi}{4} \leq x \leq \frac{5\pi}{4}, \cos(x) \leq y \leq \sin(x) \right\}$.

(Hinweis: $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)

$$\begin{aligned} &\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\int_{\cos(x)}^{\sin(x)} (1 - 2x - 3y) dy \right) dx \\ &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(y - 2xy - \frac{3y^2}{2} \right) \Big|_{\cos(x)}^{\sin(x)} dx \\ &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left((1 - 2x) \sin(x) + (-1 + 2x) \cos(x) - \frac{3(\sin(x)^2 - \cos(x)^2)}{2} \right) dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left((1-2x)\sin(x) + (-1+2x)\cos(x) - \frac{3(1-2\cos(x)^2)}{2} \right) dx \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left((1-2x)\sin(x) + (-1+2x)\cos(x) + 3\cos(x)^2 - \frac{3}{2} \right) dx \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x) \cdot \sin(x) dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x) \cdot \cos(x) dx + 3 \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2 dx - \frac{3}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 dx
\end{aligned}$$

Berechnung der Integrale:

(Man muss dabei beachten, dass die Grenzen jeweils um π verschieden sind, sich also bei den Funktionswerten das Vorzeichen ändert)

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x)\sin(x) dx &= -(1-2x)\cos(x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - 2 \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x) dx \\
&= \left(\left(\frac{(2-5\pi)\sqrt{2}}{4} \right) - \left(\frac{(\pi-2)\sqrt{2}}{4} \right) \right) - 2 \sin(x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \left(\frac{4\sqrt{2} - 6\pi\sqrt{2}}{4} \right) - 2 \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \\
&= \sqrt{2} - \frac{3\pi\sqrt{2}}{2} + 2\sqrt{2} \\
&= \left(3 - \frac{3\pi}{2} \right) \cdot \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x) \cdot \cos(x) dx &= (-1+2x) \cdot \sin(x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + 2 \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x) dx \\
&= \left(\left(\frac{(2-5\pi)\sqrt{2}}{4} \right) - \left(\frac{(-2+\pi)\sqrt{2}}{4} \right) \right) - 2 \cos(x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \left(\frac{(2-5\pi) \cdot \sqrt{2} + (2-\pi) \cdot \sqrt{2}}{4} \right) - 2 \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \\
&= \sqrt{2} - \frac{6\pi\sqrt{2}}{4} - 2\sqrt{2} \\
&= \left(-1 - \frac{3\pi}{2} \right) \cdot \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
3 \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2 dx &= \frac{3}{2} \cdot \left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(2x) dx \right) \\
&= \frac{3}{2} \cdot \left(x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \frac{\cos(2x)}{2} \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \right) \\
&= \frac{3}{2} \cdot \left(\pi - \frac{1}{2} \cdot \left(\frac{\cos\left(\frac{5\pi}{2}\right)}{2} - \frac{\cos\left(\frac{\pi}{2}\right)}{2} \right) \right) \\
&= \frac{3\pi}{2} \\
-\frac{3}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 dx &= -\frac{3}{2} x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\frac{3}{2} \cdot \left(\frac{5\pi}{4} - \frac{\pi}{4} \right) = -\frac{3\pi}{2}
\end{aligned}$$

Einsetzen der Integrale:

$$\begin{aligned}
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x) \cdot \sin(x) dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x) \cdot \cos(x) dx + 3 \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2 dx - \frac{3}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 dx \\
&= \left(3 - \frac{3\pi}{2} \right) \cdot \sqrt{2} + \left(-1 - \frac{3\pi}{2} \right) \cdot \sqrt{2} + \frac{3\pi}{2} - \frac{3\pi}{2} \\
&= (2 - 3\pi) \cdot \sqrt{2}
\end{aligned}$$

- 3) Sei $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, x \leq 0, y \geq 0\}$. Berechnen Sie das Gebietsintegral

$$\iint_D \cos(x^2 + y^2) dx dy$$

durch Transformation auf Polarkoordinaten.

Transformation:

$$x = r \cdot \cos(\varphi), \quad y = r \cdot \sin(\varphi)$$

$$x^2 + y^2 \leq 1 \Rightarrow 0 \leq r^2 \leq 1$$

$$x \leq 0, y \geq 0 \Rightarrow \frac{\pi}{2} \leq \varphi \leq \pi$$

$$\begin{aligned}
\iint_D \cos(x^2 + y^2) dx dy &= \iint_D \cos\left(r^2 \cdot (\cos(x)^2 + \sin(x)^2)\right) dx dy \\
&= \int_0^1 \left(\int_{\frac{\pi}{2}}^{\pi} \cos(r^2) \cdot r d\varphi \right) dr \\
&= \int_0^1 \cos(r^2) \cdot r \cdot \varphi \Big|_{\frac{\pi}{2}}^{\pi} dr = \frac{\pi}{2} \int_0^1 \cos(r^2) \cdot r dr
\end{aligned}$$

SUB:

$$\begin{aligned}
t = r^2 &\Leftrightarrow \frac{dt}{dr} = 2r \Leftrightarrow dr = \frac{dt}{2r} \\
\Rightarrow \frac{\pi}{2} \int_0^1 \cos(t) \cdot r \frac{dt}{2r} &= \frac{\pi}{4} \int_0^1 \cos(t) dt \\
&= \frac{\pi}{4} \sin(t) \Big|_0^1 \\
&= \frac{\pi}{4} \sin(1)
\end{aligned}$$

- 4) Berechnen Sie $\iint_D 4x dx dy$. Dabei sei D der Bereich, der durch $y = \ln(x)$, $y = 0$ und $x = 2$ eingeschlossen wird.

$$\Rightarrow D := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2, 0 \leq y \leq \ln(x)\}$$

$$\begin{aligned}
\int_1^2 \left(\int_0^{\ln(x)} 4x dy \right) dx &= \int_1^2 4xy \Big|_0^{\ln(x)} dx \\
&= 4 \int_1^2 x \cdot \ln(x) dx \\
&= 4 \left(\frac{x^2 \cdot \ln(x)}{2} \Big|_1^2 - \frac{1}{2} \int_1^2 x dx \right) \\
&= 4 \left(2 \cdot \ln(2) - \frac{x^2}{4} \Big|_1^2 \right) \\
&= 8 \cdot \ln(2) - 3
\end{aligned}$$