Tutorium Mathe 2 MT

Aufgabenblatt: Integralrechnung / Gebietsintegrale (+ Lösungen)

1) Berechnen Sie das Gebietsintegral $\iint\limits_{0\le y\le x\le 1} \left(x^2+y^2\right)dxdy$.

$$\int_0^1\left(\int_0^x x^2+y^2dy\right)dx=\int_0^1 x^2y+\frac{y^3}{3}\bigg|_0^x dx$$
$$=\frac{4}{3}\int_0^1 x^3dx$$
$$=\frac{4}{3}\cdot\frac{x^4}{4}\bigg|_0^1$$
$$=\frac{4}{3}\cdot\frac{1}{4}$$
$$=\frac{1}{3}$$

2) Berechnen Sie das Integral

$$\iint\limits_D (1-2x-3y)\,dxdy$$

über dem Gebiet $D:=\left\{(x,y)\in\mathbb{R}^2 \mid \frac{\pi}{4}\le x\le\frac{5\pi}{4},\ \cos(x)\le y\le\sin(x)\right\}$.

(Hinweis: $\sin\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$)

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(\int_{\cos(x)}^{\sin(x)}(1-2x-3y)\,dy\right)dx$$
$$=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(y-2xy-\frac{3y^2}{2}\right)\bigg|_{\cos(x)}^{\sin(x)}dx$$
$$=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left((1-2x)\sin(x)+(-1+2x)\cos(x)-\frac{3\left(\sin(x)^2-\cos(x)^2\right)}{2}\right)dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left((1-2x)\sin(x) + (-1+2x)\cos(x) - \frac{3\left(1-2\cos(x)^2\right)}{2} \right) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left((1-2x)\sin(x) + (-1+2x)\cos(x) + 3\cos(x)^2 - \frac{3}{2} \right) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x)\cdot\sin(x)\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x)\cdot\cos(x)\,dx + 3\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2\,dx - \frac{3}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1\,dx$$

Berechnung der Integrale:
(Man muss dabei beachten, dass die Grenzen jeweils um π verschieden sind, sich also bei den Funktionswerten das Vorzeichen ändert)

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x)\sin(x)\,dx = -(1-2x)\cos(x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)\,dx$$

$$= \left(\left(\frac{(2-5\pi)\sqrt{2}}{4} \right) - \left(\frac{(\pi-2)\sqrt{2}}{4} \right) \right) - 2\sin(x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left(\frac{4\sqrt{2} - 6\pi\sqrt{2}}{4} \right) - 2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)$$

$$= \sqrt{2} - \frac{3\pi\sqrt{2}}{2} + 2\sqrt{2}$$

$$= \left(3 - \frac{3\pi}{2} \right)\cdot\sqrt{2}$$

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x)\cdot\cos(x)\,dx = (-1+2x)\cdot\sin(x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x)\,dx$$

$$= \left(\left(\frac{(2-5\pi)\sqrt{2}}{4} \right) - \left(\frac{(-2+\pi)\sqrt{2}}{4} \right) \right) - 2\cos(x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left(\frac{(2-5\pi)\cdot\sqrt{2} + (2-\pi)\cdot\sqrt{2}}{4} \right) - 2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)$$

$$= \sqrt{2} - \frac{6\pi\sqrt{2}}{4} - 2\sqrt{2}$$

$$= \left(-1 - \frac{3\pi}{2} \right)\cdot\sqrt{2}$$

$$3\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2\,dx = \frac{3}{2}\cdot\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1\,dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(2x)\,dx\right)$$

$$= \frac{3}{2}\cdot\left(x\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \frac{\cos(2x)}{2}\Bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \right)$$

$$= \frac{3}{2}\cdot\left(\pi - \frac{1}{2}\cdot\left(\frac{\cos\left(\frac{5\pi}{2}\right)}{2} - \frac{\cos\left(\frac{\pi}{2}\right)}{2}\right)\right)$$

$$= \frac{3\pi}{2}$$

$$-\frac{3}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1\,dx = -\frac{3}{2}x\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\frac{3}{2}\cdot\left(\frac{5\pi}{4} - \frac{\pi}{4}\right) = -\frac{3\pi}{2}$$

Einsetzen der Integrale:

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-2x)\cdot\sin(x)\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-1+2x)\cdot\cos(x)\,dx + 3\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)^2\,dx - \frac{3}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1\,dx$$

$$= \left(3 - \frac{3\pi}{2}\right)\cdot\sqrt{2} + \left(-1 - \frac{3\pi}{2}\right)\cdot\sqrt{2} + \frac{3\pi}{2} - \frac{3\pi}{2}$$

$$= (2-3\pi)\cdot\sqrt{2}$$

3) Sei $D := \left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1,\ x\le 0,\ y\ge 0\right\}$. Berechnen Sie das Gebietsintegral

$$\iint_D \cos\left(x^2+y^2\right)dxdy$$

durch Transformation auf Polarkoordinaten.

Transformation:

$$x = r\cdot\cos(\varphi),\ y = r\cdot\sin(\varphi)$$

$$x^2+y^2 \le 1 \Rightarrow 0 \le r^2 \le 1$$

$$x\le 0, y\ge 0 \Rightarrow \frac{\pi}{2} \le \varphi \le \pi$$

$$\iint_D \cos\left(x^2+y^2\right)dxdy = \iint_D \cos\left(r^2\cdot\left(\cos\left(x\right)^2+\sin\left(x\right)^2\right)\right)dxdy$$

$$= \int_0^1 \left(\int_{\frac{\pi}{2}}^{\pi} \cos\left(r^2\right)\cdot r\,d\varphi \right) dr$$

$$= \int_0^1 \cos\left(r^2\right)\cdot r\cdot\varphi\Big|_{\frac{\pi}{2}}^{\pi} dr = \frac{\pi}{2}\int_0^1 \cos\left(r^2\right)\cdot r\,dr$$

$SUB:$

$$t = r^2 \Leftrightarrow \frac{dt}{dr} = 2r \Leftrightarrow dr = \frac{dt}{2r}$$

$$\Rightarrow \frac{\pi}{2}\int_0^1 \cos\left(t\right)\cdot r\frac{dt}{2r} = \frac{\pi}{4}\int_0^1 \cos\left(t\right)dt$$

$$= \frac{\pi}{4}\sin\left(t\right)\Big|_0^1$$

$$= \frac{\pi}{4}\sin\left(1\right)$$

4) Berechnen Sie $\iint_D 4xdxdy$. Dabei sei D der Bereich, der durch $y=\ln\left(x\right)$, $y=0$ und $x=2$ eingeschlossen wird.

$$\Rightarrow D := \left\{\left(x,y\right)\in\mathbb{R}^2 \mid 1\le x\le 2,\ 0\le y\le \ln\left(x\right)\right\}$$

$$\int_1^2\left(\int_0^{\ln(x)} 4x\,dy\right)dx = \int_1^2 4xy\Big|_0^{\ln(x)} dx$$

$$= 4\int_1^2 x\cdot\ln\left(x\right)dx$$

$$= 4\left(\frac{x^2\cdot\ln\left(x\right)}{2}\Big|_1^2 - \frac{1}{2}\int_1^2 x\,dx\right)$$

$$= 4\left(2\cdot\ln\left(2\right) - \frac{x^2}{4}\Big|_1^2\right)$$

$$= 8\cdot\ln\left(2\right) - 3$$